103. —CONTINUED—

Let $g(x) = \frac{1}{2}\left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right)$.

$$g'(x) = \frac{1}{2}\left(x\frac{1}{2}(x^2+1)^{-1/2}(2x) + \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2 + (x^2+1) + 1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{2(x^2+1)}{\sqrt{x^2+1}}\right) = \sqrt{x^2+1}$$

Thus, $\int \sqrt{x^2+1}\,dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right) + C.$

104. Let $I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx.$

I is defined and continuous on $[2, 4]$. Note the symmetry: as x goes from 2 to 4, $9 - x$ goes from 7 to 5 and $x + 3$ goes from 5 to 7. So, let $y = 6 - x$, $dy = -dx$.

$$I = \int_4^2 \frac{\sqrt{\ln(3+y)}}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}}(-dy) = \int_2^4 \frac{\sqrt{\ln(3+y)}}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}}\,dy$$

Adding:

$$2I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx + \int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(3+x)} + \sqrt{\ln(9-x)}}\,dx = \int_2^4 dx = 2 \Rightarrow I = 1$$

You can easily check this result numerically.

Section 8.2 Integration by Parts

1. $\frac{d}{dx}[\sin x - x\cos x] = \cos x - (-x\sin x + \cos x) = x\sin x$

Matches (b)

2. $\frac{d}{dx}[x^2\sin x + 2x\cos x - 2\sin x] = x^2\cos x + 2x\sin x - 2x\sin x + 2\cos x - 2\cos x = x^2\cos x$

Matches (d)

3. $\frac{d}{dx}[x^2e^x - 2xe^x + 2e^x] = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x$

$= x^2e^x$

Matches (c)

4. $\frac{d}{dx}[-x + x\ln x] = -1 + x\left(\frac{1}{x}\right) + \ln x = \ln x$

Matches (a)

5. $\int xe^{2x}\,dx$

$u = x,\ dv = e^{2x}\,dx$

6. $\int x^2e^{2x}\,dx$

$u = x^2,\ dv = e^{2x}\,dx$

7. $\int (\ln x)^2\,dx$

$u = (\ln x)^2,\ dv = dx$

8. $\int \ln 3x\, dx$

$u = \ln 3x,\ dv = dx$

9. $\int x \sec^2 x\, dx$

$u = x,\ dv = \sec^2 x\, dx$

10. $\int x^2 \cos x\, dx$

$u = x^2,\ dv = \cos x\, dx$

11. $dv = e^{-2x}\, dx \implies v = \int e^{-2x}\, dx = -\frac{1}{2}e^{-2x}$

$u = x \implies du = dx$

$$\int xe^{-2x}\, dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\, dx$$
$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$
$$= \frac{-1}{4e^{2x}}(2x + 1) + C$$

12. $dv = e^{-x}\, dx \implies v = \int e^{-x}\, dx = -e^{-x}$

$u = x \implies du = dx$

$$2\int \frac{x}{e^x}\, dx = 2\int xe^{-x}\, dx$$
$$= 2\left[-xe^{-x} - \int -e^{-x}\, dx\right]$$
$$= 2[-xe^{-x} - e^{-x}] + C$$
$$= -2xe^{-x} - 2e^{-x} + C$$

13. Use integration by parts three times.

(1) $dv = e^x\, dx \implies v = \int e^x\, dx = e^x$

$u = x^3 \implies du = 3x^2\, dx$

(2) $dv = e^x\, dx \implies v = \int e^x\, dx = e^x$

$u = x^2 \implies du = 2x\, dx$

(3) $dv = e^x\, dx \implies v = \int e^x\, dx = e^x$

$u = x \implies du = dx$

$$\int x^3e^x\, dx = x^3e^x - 3\int x^2e^x\, dx = x^3e^x - 3x^2e^x + 6\int xe^x\, dx$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C$$

14. $\displaystyle\int \frac{e^{1/t}}{t^2}\, dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right) dt = -e^{1/t} + C$

15. $\displaystyle\int x^2e^{x^3}\, dx = \frac{1}{3}\int e^{x^3}(3x^2)\, dx = \frac{1}{3}e^{x^3} + C$

16. $dv = x^4\, dx \implies v = \frac{x^5}{5}$

$u = \ln x \implies du = \frac{1}{x}\, dx$

$$\int x^4 \ln x\, dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\left(\frac{1}{x}\right) dx$$
$$= \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4\, dx$$
$$= \frac{x^5}{5}\ln x - \frac{1}{25}x^5 + C$$
$$= \frac{x^5}{25}(5\ln x - 1) + C$$

17. $dv = t\, dt \implies v = \int t\, dt = \frac{t^2}{2}$

$u = \ln(t + 1) \implies du = \frac{1}{t + 1}\, dt$

$$\int t\ln(t + 1)\, dt = \frac{t^2}{2}\ln(t + 1) - \frac{1}{2}\int \frac{t^2}{t + 1}\, dt$$
$$= \frac{t^2}{2}\ln(t + 1) - \frac{1}{2}\int\left(t - 1 + \frac{1}{t + 1}\right) dt$$
$$= \frac{t^2}{2}\ln(t + 1) - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t + 1)\right] + C$$
$$= \frac{1}{4}[2(t^2 - 1)\ln|t + 1| - t^2 + 2t] + C$$

18. Let $u = \ln x$, $du = \frac{1}{x}\, dx$.

$$\int \frac{1}{x(\ln x)^3}\, dx = \int (\ln x)^{-3}\left(\frac{1}{x}\right) dx = \frac{-1}{2(\ln x)^2} + C$$

19. Let $u = \ln x$, $du = \frac{1}{x}\, dx$.

$$\int \frac{(\ln x)^2}{x}\, dx = \int (\ln x)^2\left(\frac{1}{x}\right) dx = \frac{(\ln x)^3}{3} + C$$

20. $dv = \frac{1}{x^2}\,dx \Rightarrow \quad v = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$u = \ln x \quad \Rightarrow du = \frac{1}{x}\,dx$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

21. $dv = \frac{1}{(2x+1)^2}\,dx \Rightarrow \quad v = \int (2x+1)^{-2}\,d$

$= -\frac{1}{2(2x+1)}$

$u = xe^{2x} \quad \Rightarrow du = (2xe^{2x} + e^{2x})\,d$

$= e^{2x}(2x+1)\,dx$

$$\int \frac{xe^{2x}}{(2x+1)^2}\,dx = -\frac{xe^{2x}}{2(2x+1)} + \int \frac{e^{2x}}{2}\,dx$$

$$= \frac{-xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C = \frac{\quad}{4(}$$

22. $dv = \frac{x}{(x^2+1)^2}\,dx \Rightarrow \quad v = \int (x^2+1)^{-2}x\,dx = -\frac{1}{2(x^2+1)}$

$u = x^2e^{x^2} \quad \Rightarrow du = (2x^3e^{x^2} + 2xe^{x^2})\,dx = 2xe^{x^2}(x^2+1)\,dx$

$$\int \frac{x^3e^{x^2}}{(x^2+1)^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \int xe^{x^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2(x^2+1)} + C$$

23. Use integration by parts twice.

(1) $dv = e^x\,dx \Rightarrow \quad v = \int e^x\,dx = e^x$

$u = x^2 \quad \Rightarrow du = 2x\,dx$

(2) $dv = e^x\,dx \Rightarrow \quad v = \int e^x\,dx = e^x$

$u = x \quad \Rightarrow du = dx$

$$\int (x^2-1)e^x\,dx = \int x^2e^x\,dx - \int e^x\,dx = x^2e^x - 2\int xe^x\,dx - e^x$$

$$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right] - e^x = x^2e^x - 2xe^x + e^x + C = (x-1)^2e^x + C$$

24. $dv = \frac{1}{x^2}\,dx \Rightarrow \quad v = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$u = \ln 2x \Rightarrow du = \frac{1}{x}\,dx$

$$\int \frac{\ln(2x)}{x^2}\,dx = -\frac{\ln(2x)}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln(2x)}{x} - \frac{1}{x} + C$$

$$= -\frac{\ln(2x) + 1}{x} + C$$

25. $dv = \sqrt{x-1}\,dx \Rightarrow \quad v = \int (x-1)^{1/2}\,dx = \frac{2}{3}(x$

$u = x \quad \Rightarrow du = dx$

$$\int x\sqrt{x-1}\,dx = \frac{2}{3}x(x-1)^{3/2} - \frac{2}{3}\int (x-1)^{3/2}\,dx$$

$$= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C$$

$$= \frac{2(x-1)^{3/2}}{15}(3x+2) + C$$

26. $dv = \frac{1}{\sqrt{2+3x}}\,dx \Rightarrow \quad v = \int (2+3x)^{-1/2}\,dx = \frac{2}{3}\sqrt{2+3x}$

$u = x \quad \Rightarrow du = dx$

$$\int \frac{x}{\sqrt{2+3x}}\,dx = \frac{2x\sqrt{2+3x}}{3} - \frac{2}{3}\int \sqrt{2+3x}\,dx$$

$$= \frac{2x\sqrt{2+3x}}{3} - \frac{4}{27}(2+3x)^{3/2} + C = \frac{2\sqrt{2+3x}}{27}[9x - 2(2+3x)] + C = \frac{2\sqrt{2+3x}}{27}(3x-4) + C$$

27. $dv = \cos x\,dx \Rightarrow v = \int \cos x\,dx = \sin x$

$u = x \Rightarrow du = dx$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

28. $dv = \sin x\,dx \Rightarrow v = -\cos x$

$u = x \Rightarrow du = dx$

$$\int x\sin dx = -x\cos x - \int -\cos x\,dx$$
$$= -x\cos x + \sin x + C$$

29. Use integration by parts three times.

(1) $u = x^3, du = 3x^2\,dx, dv = \sin x\,dx, v = -\cos x$

$$\int x^3 \sin dx = -x^3\cos x + 3\int x^2\cos x\,dx$$

(2) $u = x^2, du = 2x\,dx, dv = \cos x\,dx, v = \sin x$

$$\int x^3\sin x\,dx = -x^3\cos x + 3\left[x^2\sin x - 2\int x\sin x\,dx\right]$$
$$= -x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx$$

(3) $u = x, du = dx, dv = \sin x\,dx, v = -\cos x$

$$\int x^3\sin x\,dx = -x^3\cos x + 3x^2\sin x - 6\left[-x\cos x + \int\cos x\,dx\right]$$
$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

30. Use integration by parts twice.

(1) $u = x^2, du = 2x\,dx, dv = \cos x\,dx, v = \sin x$

$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$$

(2) $u = x, du = dx, dv = \sin x\,dx, v = -\cos x$

$$\int x^2\cos x\,dx = x^2\sin x - 2\left[-x\cos x + \int\cos x\,dx\right]$$
$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

31. $u = t, du = dt, dv = \csc t\cot dt, v = -\csc t$

$$\int t\csc t\cot t\,dt = -t\csc t + \int\csc t\,dt$$
$$= -t\csc t - \ln|\csc t + \cot t| + C$$

32. $dv = \sec\theta\tan\theta\,d\theta \Rightarrow v = \int\sec\theta\tan\theta\,d\theta = \sec\theta$

$u = \theta \Rightarrow du = d\theta$

$$\int\theta\sec\theta\tan\theta\,d\theta = \theta\sec\theta - \int\sec\theta\,d\theta$$
$$= \theta\sec\theta - \ln|\sec\theta + \tan\theta| + C$$

33. $dv = dx \Rightarrow v = \int dx = x$

$u = \arctan x \Rightarrow du = \dfrac{1}{1+x^2}\,dx$

$$\int\arctan x\,dx = x\arctan x - \int\frac{x}{1+x^2}\,dx$$
$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

34. $dv = dx \Rightarrow v = \int dx = x$

$u = \arccos x \Rightarrow du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$

$$4\int\arccos x\,dx = 4\left[x\arccos x + \int\frac{x}{\sqrt{1-x^2}}\,dx\right]$$
$$= 4\left[x\arccos x - \sqrt{1-x^2}\right] + C$$

35. Use integration by parts twice.

(1) $dv = e^{2x}\,dx \Rightarrow v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$u = \sin x \Rightarrow du = \cos x\,dx$

(2) $dv = e^{2x}\,dx \Rightarrow v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$u = \cos x \Rightarrow du = -\sin x\,dx$

—CONTINUED—

35. —CONTINUED—

$$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left(\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx\right)$$

$$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$$

36. Use integration by parts twice.

(1) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = \cos 2x \Rightarrow du = -2\sin 2x\,dx$

(2) $dv = e^x\,dx \Rightarrow v = \int e^x\,dx = e^x$

$u = \sin 2x \Rightarrow du = 2\cos 2x\,dx$

$$\int e^x\cos 2x\,dx = e^x\cos 2x + 2\int e^x\sin 2x\,dx = e^x\cos 2x + 2\left(e^x\sin 2x - 2\int e^x\cos 2x\,dx\right)$$

$$5\int e^x\cos 2x\,dx = e^x\cos 2x + 2e^x\sin 2x$$

$$\int e^x\cos 2x\,dx = \frac{e^x}{5}(\cos 2x + 2\sin 2x) + C$$

37. $y' = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

38. $dv = dx \Rightarrow v = x$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$y' = \ln x$

$$y = \int \ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx$$

$$= x\ln x - x + C = x(-1 + \ln x) + C$$

39. Use integration by parts twice.

(1) $dv = \frac{1}{\sqrt{2+3t}}\,dt \Rightarrow v = \int (2+3t)^{-1/2}\,dt = \frac{2}{3}\sqrt{2+3t}$

$u = t^2 \Rightarrow du = 2t\,dt$

(2) $dv = \sqrt{2+3t}\,dt \Rightarrow v = \int (2+3t)^{1/2}\,dt = \frac{2}{9}(2+3t)^{3/2}$

$u = t \Rightarrow du = dt$

$$y = \int \frac{t^2}{\sqrt{2+3t}}\,dt = \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3}\int t\sqrt{2+3t}\,dt$$

$$= \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3}\left[\frac{2t}{9}(2+3t)^{3/2} - \frac{2}{9}\int (2+3t)^{3/2}\,dt\right]$$

$$= \frac{2t^2\sqrt{2+3t}}{3} - \frac{8t}{27}(2+3t)^{3/2} + \frac{16}{405}(2+3t)^{5/2} + C$$

$$= \frac{2\sqrt{2+3t}}{405}(27t^2 - 24t + 32) + C$$

40. Use integration by parts twice.

(1) $dv = \sqrt{x-1}\,dx \Rightarrow v = \int (x-1)^{1/2}\,dx = \frac{2}{3}(x-1)^{3/2}$

$u = x^2 \Rightarrow du = 2x\,dx$

(2) $dv = (x-1)^{3/2}\,dx \Rightarrow v = \int (x-1)^{3/2}\,dx = \frac{2}{5}(x-1)^{5/2}$

$u = x \Rightarrow du = dx$

$$y = \int x^2\sqrt{x-1}\,dx$$

$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\int x(x-1)^{3/2}\,dx = \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\left[\frac{2}{5}x(x-1)^{5/2} - \frac{2}{5}\int (x-1)^{5/2}\,dx\right]$$

$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C = \frac{2(x-1)^{3/2}}{105}(15x^2 + 12x + 8) + C$$

41. $(\cos y)y' = 2x$

$$\int \cos y\,dy = \int 2x\,dx$$

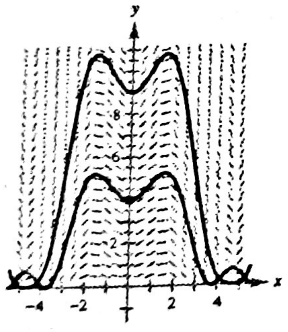
$$\sin y = x^2 + C$$

42. $dv = dx \Rightarrow v = \int dx = x$

$$u = \arctan\frac{x}{2} \Rightarrow du = \frac{1}{1 + (x/2)^2}\left(\frac{1}{2}\right)dx = \frac{2}{4 + x^2}\,dx$$

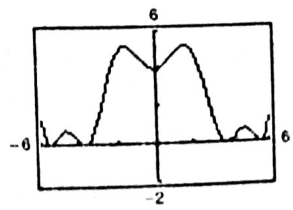
$$y = \int \arctan\frac{x}{2}\,dx = x\arctan\frac{x}{2} - \int \frac{2x}{4 + x^2}\,dx = x\arctan\frac{x}{2} - \ln(4 + x^2) + C$$

43. (a)



(b)

$$\frac{dy}{dx} = x\sqrt{y}\cos x, \quad (0, 4)$$



$$\int \frac{dy}{\sqrt{y}} = \int x\cos x\,dx$$

$$\int y^{-1/2}\,dy = \int x\cos x\,dx \qquad (u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x)$$

$$2y^{1/2} = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

$(0, 4)$: $2(4)^{1/2} = 0 + 1 + C \Rightarrow C = 3$

$$2\sqrt{y} = x\sin x + \cos x + 3$$